JOURNAL OF COMPUTATIONAL PHYSICS 63, 191–200 (1986)

An Embedding Method for the Steady Euler Equations

SHIH-HUNG CHANG*

Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115

AND

GARY M. JOHNSON†

Institute for Computational Studies, P. O. Box 1852, Fort Collins, Colorado 80522

Received August 2, 1983; revised December 27, 1984

A recent approach to the numerical solution of the steady Euler equations is to embed the first-order Euler system in a second-order system and then to obtain the solution of the original system by solving the embedded one together with certain additional boundary conditions. Initial development of this approach and computational experimentation with it have been based on heuristic physical reasoning. In this paper the theoretical justification for the embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler equations will not produce extraneous solutions. © 1986 Academic Press, Inc.

1. INTRODUCTION

In the development of numerical solution procedures for the steady Euler equations, the common approach is to replace the steady equations by their unsteady counterparts and then to seek a temporally asymptotic steady solution, either in real time [1, 2] or in pseudo time [3–5]. Due to the difficulties associated with the numerical solution of a direct finite difference representation of the steady Euler equations, relatively few departures from this approach are to be found in the literature. Steger and Lomax [6] developed an iterative procedure for solving a nonconservation form of the steady Euler equations for subcritical flow with small shear. Desideri and Lomax [7] investigated preconditioning procedures on the matrix system arising from the finite differencing of the Euler equations. Bruneau, Chattot, Laminie, and Guiu-Roux [8] have used a variational approach to transform the Euler equations into an equivalent second-order system. Preliminary

* Work supported in part by the NASA Lewis Research Center under grant NAG3-339.
† Work performed in part while the author was a staff member at the NASA Lewis Research Center.

0021-9991/86 $3.00

numerical results have been presented for two-dimensional internal flows. Jesperson [9] has recently adapted multigrid techniques to the solution of the Euler equations and has presented results for transonic flows over airfoils.

Johnson [10–12] proposed a surrogate-equation technique, in which the first-order steady Euler equations are embedded in a certain second-order system of equations. The solution of the original Euler equations is then obtained by solving this second-order system together with some additional boundary conditions. The advantages of such an approach are that the difficulties of solving the direct difference representation of the steady Euler equations can be bypassed and the resulting second-order embedded system can be solved by a variety of well-proven numerical procedures. Initial development of this approach and computational experimentation with it have been based on heuristic physical reasoning. This has led to the construction of a relaxation procedure for the solution of two-dimensional steady flow problems.

In this paper the theoretical justification for such an embedding approach is addressed. It is proven that, with the appropriate choice of embedding operator and additional boundary conditions, the solution to the embedded system is exactly the one to the original Euler equations. Hence, solving the embedded version of the Euler equations will not produce extraneous solutions. The following section contains the main theorem and proof for the two-dimensional Euler equations. In Section 3 we show that for the Cauchy–Riemann equations a similar result follows immediately from the main theorem. Sections 4 and 5 contain remarks on implementation and conclusions.

2. Embedding Theorem

The steady Euler equations can be written in vector form as

$$f_x + g_y = 0, \tag{1}$$

where x and y are Cartesian coordinates,

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix}.$$

Here ρ, p, u, v, and E denote, respectively, the density, static pressure, velocity components in the x and y directions, and the total energy per unit volume. Furthermore

$$E = \rho[e + \tfrac{1}{2}(u^2 + v^2)],$$

where the specific internal energy e is related to the pressure and density by the gas law

$$p = (\gamma - 1)\rho e$$

with γ denoting the ratio of specific heats.

Let

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}.$$

By Euler's theorem on homogeneous functions, f and g can be expressed (see, e.g., [13, 14]) as $f = Aw$ and $G = Bw$, where A and B are the Jacobian matrices

$$A \equiv \frac{\partial f}{\partial w} \quad \text{and} \quad B \equiv \frac{\partial g}{\partial w}.$$

We have

$$A = -\begin{bmatrix} 0 & -1 & 0 & 0 \\ \frac{3-\gamma}{2} u^2 + \frac{1-\gamma}{2} v^2 & (\gamma - 3)u & (\gamma - 1)v & 1-\gamma \\ uv & -v & -u & 0 \\ \frac{\gamma E u}{\rho} + (1-\gamma)u(u^2+v^2) & -\frac{\gamma E}{\rho} + \frac{\gamma - 1}{2}(3u^2+v^2) & (\gamma - 1)uv & -\gamma u \end{bmatrix}$$

and

$$B = -\begin{bmatrix} 0 & 0 & -1 & 0 \\ uv & -v & -u & 0 \\ \frac{3-\gamma}{2} v^2 + \frac{1-\gamma}{2} u^2 & (\gamma - 1)u & (\gamma - 3)v & 1-\gamma \\ \frac{\gamma v E}{\rho} + (1-\gamma)v(u^2+v^2) & (\gamma - 1)uv & \frac{-\gamma E}{\rho} + \frac{\gamma - 1}{2}(3v^2+u^2) & -\gamma v \end{bmatrix}.$$

Now, Eq. (1) can be written as

$$\frac{\partial}{\partial x}(Aw) + \frac{\partial}{\partial y}(Bw) = 0$$

or

$$\left[\frac{\partial}{\partial x}(A \quad) + \frac{\partial}{\partial y}(B \quad)\right] w = 0. \tag{2}$$

Let L denote the differential operator

$$L = \frac{\partial}{\partial x}(A \quad) + \frac{\partial}{\partial y}(B \quad). \tag{3}$$

Then the Euler equations become

$$Lw = 0. \tag{4}$$

Now, let L^* be the formal adjoint operator to L defined by

$$L^* = -\left(A^{\mathrm{T}} \frac{\partial}{\partial x} + B^{\mathrm{T}} \frac{\partial}{\partial y}\right), \tag{5}$$

where A^{T} and B^{T} are the transposes of A and B, respectively. We may then consider the Euler equations (4) as embedded in the second-order system

$$L^*Lw = 0. \tag{6}$$

Let D be a bounded closed region with a piecewise smooth boundary ∂D. Figure 1 shows a typical computational domain for internal flow problems. For simplicity of argument, assume that Eq. (6) is defined in a domain containing D. We shall show that with an additional condition on the boundary, solutions of Eq. (6) are also solutions of Eq. (4) in D. We first establish the following lemma which is needed later in the proof of our theorem.

LEMMA. *Let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product in four-dimensional space. Then for any w admissible to L^*Lw, we have*

$$\langle Lw, Lw \rangle = \langle w, L^*Lw \rangle + \frac{\partial}{\partial x}\langle Aw, Lw \rangle + \frac{\partial}{\partial y}\langle Bw, Lw \rangle.$$

Proof. For any differentiable vector-valued functions U and V, we have

$$\left\langle \frac{\partial U}{\partial x}, V \right\rangle = -\left\langle U, \frac{\partial V}{\partial x} \right\rangle + \frac{\partial}{\partial x}\langle U, V \rangle$$

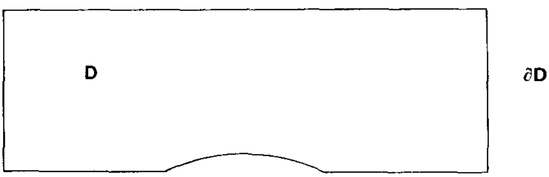

FIG. 1. A typical computational domain for internal flow problems.

and

$$\left\langle \frac{\partial U}{\partial y}, V \right\rangle = -\left\langle U, \frac{\partial V}{\partial y} \right\rangle + \frac{\partial}{\partial y} \langle U, V \rangle.$$

Hence, we have

$$\begin{aligned}
\langle Lw, Lw \rangle &= \left\langle \frac{\partial}{\partial x}(Aw) + \frac{\partial}{\partial y}(Bw), Lw \right\rangle \\
&= \left\langle \frac{\partial}{\partial x}(Aw), Lw \right\rangle + \left\langle \frac{\partial}{\partial y}(Bw), Lw \right\rangle \\
&= -\left\langle Aw, \frac{\partial}{\partial x}(Lw) \right\rangle + \frac{\partial}{\partial x} \langle Aw, Lw \rangle \\
&\quad -\left\langle Bw, \frac{\partial}{\partial y}(Lw) \right\rangle + \frac{\partial}{\partial y} \langle Bw, Lw \rangle \\
&= -\left\langle w, A^{\mathrm{T}} \frac{\partial}{\partial x}(Lw) \right\rangle + \frac{\partial}{\partial x} \langle Aw, Lw \rangle \\
&\quad -\left\langle w, B^{\mathrm{T}} \frac{\partial}{\partial y}(Lw) \right\rangle + \frac{\partial}{\partial y} \langle Bw, Lw \rangle \\
&= \left\langle w, -\left(A^{\mathrm{T}} \frac{\partial}{\partial x} + B^{\mathrm{T}} \frac{\partial}{\partial y} \right) Lw \right\rangle \\
&\quad + \frac{\partial}{\partial x} \langle Aw, Lw \rangle + \frac{\partial}{\partial y} \langle Bw, Lw \rangle \\
&= \langle w, L^* Lw \rangle + \frac{\partial}{\partial x} \langle Aw, Lw \rangle + \frac{\partial}{\partial y} \langle Bw, Lw \rangle.
\end{aligned}$$

THEOREM. *Let L, L^*, and D be defined as before and assume that the expression L^*Lw is defined in a domain containing D. If w is a solution of*

$$L^*Lw = 0 \qquad \text{in } D$$

and satisfies the additional requirement

$$Lw = 0 \qquad \text{on } \partial D,$$

then it is also a solution of

$$Lw = 0 \qquad \text{in } D.$$

Proof. By the preceding lemma, for any w admissible to L^*Lw, we have

$$\langle Lw, Lw\rangle - \langle w, L^*Lw\rangle = \frac{\partial}{\partial x}\langle Aw, Lw\rangle + \frac{\partial}{\partial y}\langle Bw, Lw\rangle.$$

Integrating over D and using Green's theorem, we obtain

$$\begin{aligned}
&\iint_D (\langle Lw, Lw\rangle - \langle w, L^*Lw\rangle)\, dx\, dy \\
&= \iint_D \left(\frac{\partial}{\partial x}\langle Aw, Lw\rangle + \frac{\partial}{\partial y}\langle Bw, Lw\rangle\right) dx\, dy \\
&= \int_{\partial D} (\langle Aw, Lw\rangle\, dy - \langle Bw, Lw\rangle\, dx).
\end{aligned} \tag{7}$$

Here the line integral in the last expression of Eq. (7) is evaluated in the counterclockwise direction over the closed contour ∂D. Now, if w satisfies the hypotheses of the theorem, i.e., $L^*Lw = 0$ in D and $Lw = 0$ on ∂D, then Eq. (7) reduces to

$$\iint_D \langle Lw, Lw\rangle\, dx\, dy = 0.$$

This implies that

$$\langle Lw, Lw\rangle = 0 \qquad \text{in } D$$

and hence

$$Lw = 0 \qquad \text{in } D.$$

Remark 1. In the above theorem, the differentiability restriction on the boundary can be replaced by the statement that the expression Lw is continuous in D and differentiable in the interior of D. Therefore, what we have established here is a theoretical justification for the embedding approach in cases where continuous solutions exist.

Remark 2. In Desideri and Lomax [7], preconditioning matrices are investigated. In our Eq. (6), L^* may be considered as a preconditioning operator. Hence, the embedding method is a preconditioning procedure for the continuous model, while Desideri and Lomax's approach is one for the corresponding discrete model.

Remark 3. The embedded system (6) and the additional boundary condition can also be derived from a least squares formulation of the Euler system (4).

The following is an immediate consequence of the above theorem. Let D^* be a domain such that D^* together with its piecewise smooth boundary, ∂D^*, is contained in D, i.e.,

$$(D^* \cup \partial D^*) \subset D.$$

Let $Sw = f$ represent certain boundary conditions associated with the original equations $Lw = 0$ in D. Assume that the boundary value problem

$$\begin{aligned} Lw &= 0 \qquad \text{in } D, \\ Sw &= f \qquad \text{on } \partial D, \end{aligned} \tag{8}$$

has a unique solution.

COROLLARY 1. *If w satisfies*

$$\begin{aligned} L^*Lw &= 0 \qquad \textit{in } D^*, \\ Lw &= 0 \qquad \textit{in } D - D^*, \\ Sw &= f \qquad \textit{on } \partial D, \end{aligned}$$

then it is the unique solution to the original boundary value problem (8).

Remark. This corollary allows one to implement the extra "boundary" condition $Lw = 0$ is a zone $D - D^*$. Since the second-order system $L^*Lw = 0$ is easier to handle than the original equations in D, it should be solved in a subdomain D^* as large as possible.

3. Cauchy–Riemann Equations

Consider the special case of the Cauchy–Riemann equations

$$u_x + v_y = 0, \tag{9}$$

$$v_x - u_y = 0. \tag{10}$$

Let

$$f = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad g = \begin{bmatrix} v \\ -u \end{bmatrix}$$

and rewrite Eqs. (9) and (10) in vector form

$$f_x + g_y = 0. \tag{11}$$

If we choose

$$w \equiv f$$

then we have

$$A = \frac{\partial f}{\partial w} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \frac{\partial g}{\partial w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Equation (11) can then be exressed as

$$\frac{\partial}{\partial x}(Aw) + \frac{\partial}{\partial y}(Bw) = 0$$

or

$$\left[\frac{\partial}{\partial x}(A\ \ ) + \frac{\partial}{\partial y}(B\ \ )\right] w = 0.$$

Hence, if we again use L to denote the differential operator

$$L = \frac{\partial}{\partial x}(A\ \ ) + \frac{\partial}{\partial y}(B\ \ ), \tag{12}$$

the Cauchy–Riemann equations can also be written as

$$Lw = 0. \tag{13}$$

Let

$$L^* = -\left(A^{\mathrm{T}} \frac{\partial}{\partial x} + B^{\mathrm{T}} \frac{\partial}{\partial y}\right). \tag{14}$$

Then Eq. (13) can be considered as embedded in

$$L^* L w = 0. \tag{15}$$

Note that a few simple matrix multiplications will reduce Eq. (15) to

$$\frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w = 0 \tag{16}$$

which demonstrates simply the well-known fact that Eqs. (9) and (10) are embedded in the second-order system (16).

Now, let D be the same region as defined previously. Then the introduction of the differential oerators L and L^* for the Cauchy–Riemann equations suggests the following immediate cosequence of the theorem in Section 2.

COROLLARY 2. *If w is a solution of Eq.* (16) *in D and if, on the boundary of D, it satisfies Eqs.* (9) *and* (10), *then it is also a solution of Eqs.* (9) *and* (10) *in D.*

Thus if one wishes to obtain the unique solution to a boundary value problem of the Cauchy–Riemann equations (9) and (10), one can also solve Eq. (16) together with the original boundary conditions and the additional requirement that Eqs. (9) and (10) be satisfied on the boundary. Phillips [15] has recently obtained a similar result for the nonhomogeneous Cauchy–Riemann equations.

4. NUMERICAL IMPLEMENTATION

The embedding technique for the full Euler equations, Eq. (4), has been successfully implemented using the mathematical formulations of the theorem in Section 2 for the case of subcritical flow in a straight channel with a 10% half-thick circular arc airfoil mounted on its lower wall (see Fig. 1). In this implementation, the grid points are placed in the cell centers and there is a fictitious layer of cells outside the true boundary. The original boundary conditions as well as the additional boundary condition, $Lw = 0$, are discretized and imposed iteratively between the fictitious grid points and the first line of true grid points inside the domain D. During the iteration process, the starting values on the fictitious points are fixed initially and all the interior values are computed using the second-order embedded system. Then the fictitious values are updated through the boundary conditions.

In solving the second-order embedded system, we have used the basic conjugate gradient method without any preconditioning. Numerical results obtained in this implementation are similar to those known results obtained by solving the unsteady equations. Other implementations and preconditionings are under investigation.

5. CONCLUSIONS

A theoretical justification has been provided for the embedding approach to the solution of the steady Euler equations. Namely, for the numerical solution of the two-dimensional steady Euler equations, it is proven that under a continuity restriction one can solve a second-order embedded system together with appropriate additional boundary conditions. This points out a more direct and potentially more efficient approach to the steady solutions than the alternative of solving the unsteady equations. The proof presented here is extendible to three dimensions and the embedding technique is applicable to a wider class of partial differential equations than the Euler equations of motion considered here.

Acknowledgments

The authors would like to thank Dr. Peter M. Sockol for helpful discussions and suggestions regarding the numerical implementations and the referees for their constructive remarks.

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